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## Large time behaviour of a generalized mean curvature flow

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This is a joint work with Professor Yoshikazu Giga of Hokkaido University and Professor Moto-Hiko Sato of Muroran Institute of Technology [GOS].

**1. Introduction.** We are interested in a motion of a hypersurface by its mean curvature with right angle boundary condition in a cylindrical domain. In particular, we would like to know how does behave the surface as time tends to infinity.

Let  $\Omega'$  be a convex bounded domain in  $\mathbf{R}^{N-1}$  with smooth boundary, where  $N \geq 2$ . We set a cylindrical domain  $\Omega := \Omega' \times \mathbf{R}$ . Suppose that  $\Omega_+(t)$  and  $\Omega_-(t)$  are open sets in  $\Omega$  at time  $t$  and  $\Omega_+(t) \cap \Omega_-(t) = \emptyset$ . We set a hypersurface  $\Gamma_t := \partial\Omega_+(t) \cap \partial\Omega_-(t) \subset \overline{\Omega}$  at time  $t$ ;  $\Gamma_t$  intersects the lateral boundary of  $\Omega$ . Let  $\mathbf{n}$  be a unit normal vector on  $\Gamma_t$  from  $\Omega_+(t)$  to  $\Omega_-(t)$ ; of course  $\mathbf{n}$  depends on time  $t$ . We consider the mean curvature flow equation

$$V = \kappa \quad \text{on} \quad \Gamma_t, \quad (1.1a)$$

$$\langle \mathbf{n}, \nu \rangle = 0 \quad \text{on} \quad b\Gamma_t := \partial\Omega \cap \Gamma_t, \quad (1.1b)$$

where  $V$  is normal velocity on  $\Gamma_t$  in the direction  $\mathbf{n}$ ,  $\kappa$  is mean curvature on  $\Gamma_t$  and  $\nu$  is an outward unit normal vector on  $\partial\Omega$ . We are interested in the behaviour of  $\Gamma_t$  as time tends to infinity. If  $\Gamma_0$  is the graph of a function on  $\Omega'$ , then there is a global-in-time graph-like smooth solution  $\Gamma_t$  of the mean curvature flow equation with right angle boundary condition starting from  $\Gamma_0$ . Moreover, the solution  $\Gamma_t$  converges to a hyperplane perpendicular to  $\partial\Omega$  in  $C^\infty$  topology. These results are due to Huisken [H]. It is interesting to study the large time behaviour of generalized interface evolution with a given initial (compact) hypersurface  $\Gamma_0$  not necessarily a graph-like surface. It is too naive to guess that the limit of  $\Gamma_t$  as  $t \rightarrow \infty$  is always a single hyperplane. Consider an initial hypersurface  $\Gamma_0$  given by  $r = r(x_N)$  where  $r$  is a distance from  $x_N$ -axis and  $\Omega'$  is a ball in  $\mathbf{R}^{N-1}$  centered at the origin. If  $r = r(x_N)$  is an even convex function, we expect that  $\Gamma_t$  pinches in a finite time

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if  $r(0)$  is very small so that  $\Gamma_0$  has a thin neck near the origin of  $\mathbf{R}^N$  provided that  $N \geq 3$ . Then it is natural to guess that  $\Gamma_t$  becomes two pieces and each piece converges to a different hyperplane. This suggests that the limit of  $\Gamma_t$  may consist of several hyperplanes perpendicular to  $\partial\Omega$ . As already pointed out in [ES]  $\Gamma_t$  may have interior even if  $\Gamma_0$  has no interior; see also [G1], [G2] for the boundary value problems and references therein. This suggests that the limit of  $\Gamma_t$  may have interior. So the best we conjecture for general initial  $\Gamma_0$  is that the limit of  $\Gamma_t$  as  $t \rightarrow \infty$  is a closed set in  $\overline{\Omega}$  and that the boundary of  $\Gamma_\infty$  consists of hyperplanes parallel to  $\Omega'$ .

To treat a hypersurface  $\Gamma_t$  we apply the level set approach as in [CGG] and [ES]. Roughly speaking, the level set approach is to regard  $\Gamma_t$  as the zero-level set of an auxiliary function  $u : (0, \infty) \times \overline{\Omega} \rightarrow \mathbf{R}$ ; say

$$\begin{aligned}\Gamma_t &= \{x \in \overline{\Omega}; u(t, x) = 0\}, \\ \Omega_\pm(t) &= \{x \in \overline{\Omega}; \pm u(t, x) > 0\}\end{aligned}$$

and each level set of  $u$  moves by (1.1a)-(1.1b). Then we obtain the level set equation of (1.1a)-(1.1b)

$$u_t - |\nabla u| \operatorname{div} (\nabla u / |\nabla u|) = 0 \quad \text{in } (0, \infty) \times \Omega, \quad (1.2a)$$

$$\partial u / \partial \nu = 0 \quad \text{on } (0, \infty) \times \partial\Omega. \quad (1.2b)$$

This is a degenerate parabolic equation. So we consider this equation in viscosity sense. This equation (1.2a)-(1.2b) was initially studied by [S] then by [GS]. They established a comparison principle to (1.2a)-(1.2b). Moreover, for each given bounded uniformly continuous function  $g$  such that

$$u(0, x) = g(x) \quad \text{on } \overline{\Omega}, \quad (1.2c)$$

they proved existence of global-in-time solution and uniqueness of solution to (1.2a)-(1.2c). Instead of studying  $\Gamma_t$  directly, we study the large time behaviour of solution of (1.2a)-(1.2c). Then we have two sub problems:

- (i) Does  $u(t, x)$  converge as  $t \rightarrow +\infty$ ?
- (ii) What is property of the limit function?

**2. Results.** Before to state our results, we have to say assumptions on  $g(x)$ .

**Assumptions on  $g$ .** We assume that  $g(x)$  is constant where  $|x_N|$  is sufficiently large; i.e., there exist constants  $c_1, c_2$  and positive constant  $m > 0$  so that

$$\begin{aligned} g(x', x_N) &= c_1 \quad \text{for all } x_N \geq m, x' \in \overline{\Omega'}, \\ g(x', x_N) &= c_2 \quad \text{for all } x_N \leq -m, x' \in \overline{\Omega'}. \end{aligned} \tag{2.1}$$

For a compact  $\Gamma_0$  this condition is not restrictive. Now we shall state our results.

**Theorem 2.1(Convergence).** *Assume that  $\Omega'$  is a smoothly bounded convex domain in  $\mathbf{R}^{N-1}$ . Assume that  $g \in C(\overline{\Omega})$  is as above. Then the unique viscosity solution  $u \in C([0, \infty) \times \overline{\Omega})$  of (1.2a)-(1.2c) satisfying (2.1) with the same  $m, c_1, c_2$  at each time converges uniformly on  $\overline{\Omega}$  to a function  $v \in C(\overline{\Omega})$  as  $t \rightarrow \infty$  that satisfies the level set minimal surface equation with the Neumann condition*

$$-|\nabla v| \operatorname{div} (\nabla v / |\nabla v|) = 0 \quad \text{in } \Omega, \tag{2.2a}$$

$$\partial v / \partial \nu = 0 \quad \text{on } \partial \Omega \tag{2.2b}$$

*in the viscosity sense. (If  $g$  is Lipschitz continuous, so is  $v$ ). Moreover,  $v$  fulfills (2.1) with the same  $m, c_1$  and  $c_2$ .*

**Remark 2.2.** The uniqueness of solution of (1.2a)-(1.2c) satisfying (2.1) is proved by the comparison theorem [S], [GS]. We take continuous functions  $g^-, g^+$  independent of  $x'$  such that

$$\begin{aligned} g^-(x) &\leq g(x) \leq g^+(x) \quad \text{on } \overline{\Omega}, \\ g^-(x) &= g(x) = g^+(x) \quad \text{for all } |x_N| \geq m, x' \in \overline{\Omega'}. \end{aligned}$$

Since  $g^-$  and  $g^+$  are stationary solution of (1.2a)-(1.2b), comparison yields  $g^- \leq u(t, \cdot) \leq g^+$  for all  $t \geq 0$ . This implies  $u$  satisfies (2.1) at each time.

**Remark 2.3.** For the Dirichlet boundary condition motion of  $\Gamma_t$  was studied by [SZ] and [ISZ] when  $\Omega$  is bounded, mean convex. The same convergence theorem was proved by [ISZ] except the statement related to (2.1).

**Remark 2.4.** The assertion is still valid for arbitrary smoothly bounded convex domain  $\Omega$  not necessarily a cylinder in  $\mathbf{R}^N$  except the statement related to (2.1). The proof goes as well as that of [ISZ].

**Theorem 2.5 (Strong maximum principle).** *Let  $\Omega'$  be a smoothly bounded domain in  $\mathbf{R}^{N-1}$ . Assume that  $v \in C(\overline{\Omega})$  is a viscosity solution of (2.2a)-(2.2b). If  $v(x', x_N)$  is a constant for sufficiently large  $x_N$  (or  $-x_N$ ), then  $v$  is independent of  $x'$  as a function in  $\overline{\Omega}$ .*

*Remark 2.6.* We cannot completely remove that  $v$  is a constant for sufficiently large  $x_N$ . If we remove this condition, we can make a counter example. Let  $N = 2$ ,  $\Omega' = (0, 1)$  and  $v(x) = x_1$ . We easily see that  $v$  is a viscosity solution of (2.2a)-(2.2b). However, each level set of  $v$  is parallel to  $x_2$ -axis. This means  $v$  is not a constant where  $x_2$  is sufficiently large.

Combining Theorems 2.1 and 2.5 we have:

**Theorem 2.7.** *Under the same hypothesis of Theorem 2.1 the solution  $u(t, x)$  converges to a function  $v = v(x_N)$  (satisfying (2.1)) uniformly in  $\overline{\Omega}$  as  $t \rightarrow \infty$ . In particular for each  $c \in \mathbf{R}$*

$$\lim_{t \rightarrow \infty} \sup \{ \text{dist}(x, \Gamma_\infty); x \in \Gamma_t \} = 0 \quad (2.3)$$

with

$$\begin{aligned} \Gamma_\infty &= \{(x', x_N) \in \mathbf{R}^N; v(x_N) = c, x' \in \overline{\Omega'}\}, \\ \Gamma_t &= \{(x', x_N) \in \mathbf{R}^N; u(t, x', x_N) = c\}, \end{aligned}$$

where  $\text{dist}(x, A) = \inf \{|x - y|; y \in A\}$ .

We conjecture that

$$\lim_{t \rightarrow \infty} \sup \{ \text{dist}(y, \Gamma_t); y \in \Gamma_\infty \} = 0. \quad (2.4)$$

We can prove (2.4) when  $\Gamma_\infty$  consists of a finite collection of parallel hyperplanes (perpendicular to  $x_N$ -axis). If (2.4) is proved, combining (2.3) and (2.4) implies that  $\Gamma_t$  converges to  $\Gamma_\infty$  in the topology of the Hausdorff distance as  $t \rightarrow \infty$ .

**3. Sketch of proof of Theorem 2.5.** To prove Theorem 2.5 we establish a kind of strong maximum principle for (2.2a)-(2.2b).

**Lemma 3.1 (Propagation of maximum, interior version).** *Let  $D'$  be a domain in  $\mathbf{R}^{N-1}$  and let  $D = D' \times (\alpha, \beta)$  with  $\alpha, \beta \in \mathbf{R}$ . Let  $w$  be an upper semicontinuous viscosity subsolution of*

$$-|\nabla w| \text{div}(\nabla w / |\nabla w|) = 0 \quad \text{in } D.$$

Assume that  $w$  attains its maximum  $K$  in  $D$ .

Let  $M \in \mathbf{R}$  be of form

$$M = \sup\{x_N \in (\alpha, \beta); w(x', x_N) = K \text{ for some } x' \in D'\}$$

If  $M < \beta$  and  $w(\cdot, M)$  attains its maximum  $K$  at some (interior) point  $\xi' \in D'$ , then  $w(x', M) = K$  for all  $x' \in D'$ .

**Lemma 3.2 (Boundary version).** Let  $D$  and  $D'$  be as in Lemma 3.1. Assume that  $\partial D'$  is  $C^2$ . Let  $w$  be an upper semicontinuous viscosity subsolution of

$$\begin{aligned} -|\nabla w| \operatorname{div}(\nabla w / |\nabla w|) &= 0 \quad \text{in } D, \\ \partial w / \partial \nu &= 0 \quad \text{on } \partial D' \times (\alpha, \beta). \end{aligned}$$

Assume that  $w$  attains its maximum  $K$  in  $\overline{D}$ . Let  $M \in \mathbf{R}$  be of form

$$M = \sup\{x_N \in (\alpha, \beta); w(x', x_N) = K \text{ for some } x' \in \overline{D'}\}.$$

If  $M < \beta$  and  $w(\cdot, M)$  attains its maximum  $K$  at some point  $\xi' \in \partial D'$ , then  $w(x', M) = K$  for all  $x' \in \overline{D'}$ .

*Sketch of proof of Theorem 2.5.* We may assume that  $v = v(x', x_N)$  is a constant  $c_1$  for sufficiently large  $x_N$ , say  $x_N \geq m$ . We set

$$A_\lambda^+ = \{x \in \overline{\Omega}; v(x) \geq \lambda\}, \quad A_\lambda^- = \{x \in \overline{\Omega}; v(x) \leq \lambda\}.$$

To show that  $v$  is independent of  $x'$ , it suffices to prove that  $A_\lambda^+$  and  $A_\lambda^-$  are perpendicular to  $x_N$ -axis for all  $\lambda > c_1$  and  $\lambda < c_1$ , respectively. Here a set  $A$  in  $\overline{\Omega}$  is called *perpendicular* to  $x_N$ -axis if  $(x', x_N) \in A$  for some  $x' \in \overline{\Omega'}$  implies  $(z, x_N) \in A$  for all  $z \in \overline{\Omega'}$ .

*Claim.* If  $A_\lambda^+$  and  $A_\lambda^-$  are perpendicular to  $x_N$ -axis for all  $\lambda > c_1$  and  $\lambda < c_1$ , respectively, then  $A_{c_1}^+$  and  $A_{c_1}^-$  are perpendicular to  $x_N$ -axis.

We can check this by contradiction. There would exist  $\hat{x}_N \in A_{c_1}^+$  such that  $v(\bar{x}', \hat{x}_N) \neq v(\bar{y}', \hat{x}_N)$  for some  $\bar{x}', \bar{y}' \in \overline{\Omega'}$  with  $\bar{x}' \neq \bar{y}'$ . We may assume that  $v(\bar{y}', \hat{x}_N) = c_1$  and we set  $\mu = v(\bar{x}', \hat{x}_N)$ . We consider the case  $\mu < c_1$ . Since  $\hat{x}_N \in A_\mu^-$  and  $\mu < c_1$ , we see that  $A_\mu^-$  is perpendicular to  $x_N$ -axis; i.e.,  $v(x', \hat{x}_N) = \mu$  for all  $x' \in \overline{\Omega'}$ . However, this contradicts that there exists  $\bar{y}'$  such that  $v(\bar{y}', \hat{x}_N) = c_1$ .

We can prove the case  $\mu > c_1$  similarly.

We shall only give a proof that  $A_\lambda^+$  is perpendicular to  $x_N$ -axis for all  $\lambda > c_1$  since the proof for  $A_\lambda^-$  is symmetric by taking  $-v$  instead of  $v$ . We may assume that  $v \leq \lambda$  on  $\bar{\Omega}$  by replacing  $v$  by  $\min(v, \lambda)$  since (1.2a)-(1.2b) is geometric so that  $\min(v, \lambda)$  is still a viscosity solution of (1.2a)-(1.2b) [CGG, S]. By these reduction it suffices to prove that

$$A_\lambda^+ = \{x \in \bar{\Omega}; v(x) = \lambda\}$$

is perpendicular to  $x_N$ -axis, when  $v \leq \lambda$  on  $\bar{\Omega}$  and  $v = c_1 < \lambda$  for  $x_N \geq m$ . We may assume that  $A_\lambda^+$  is nonempty.

Let  $\Sigma$  be the projection of  $A_\lambda^+$  on  $x_N$ -axis, i.e.,

$$\Sigma = \{x_N \in \mathbf{R}; (x', x_N) \in A_\lambda^+\}.$$

Since  $\bar{\Omega}'$  is compact and  $A_\lambda^+$  is closed by continuity of  $v$ , it is easy to see that  $\Sigma$  is a closed set in  $\mathbf{R}$ . Since  $v = c_1 < \lambda$  for  $x_N \geq m$ ,  $\Sigma$  is bounded from above. We have to take care of the case  $\Sigma$  is like a cantor set. For simplicity, we consider the case  $\Sigma$  is a bounded closed interval.

*Step 1.* At the boundary of  $\Sigma$ . If  $v(x', x_N)$  is a viscosity subsolution of (2.2a)-(2.2b) then so is  $v(x', -x_N)$ . We apply Lemmas around the maximum of  $\Sigma$  and the minimum of  $\Sigma$ . We see that  $v(x', x_N) = \lambda$  for all  $x_N \in \partial\Sigma$ ,  $x' \in \bar{\Omega}'$ .

*Step 2.* On the interior of  $\Sigma$ . There would exist a set

$$A_{-\lambda_0}^- := \{x \in \bar{\Omega}; v(x) \leq -\lambda_0\} \subset \bar{\Omega}' \times \Sigma \quad \text{with} \quad -\lambda_0 < \lambda.$$

We may assume that  $v(x) \geq -\lambda_0$  in  $\bar{\Omega}$  by replacing  $v$  by  $\max(v, -\lambda_0)$ . We set  $w(x) := -v(x)$  then  $w(x) \leq \lambda_0$  in  $\bar{\Omega}$ . We see  $w$  is a viscosity subsolution of (2.2a)-(2.2b) since  $v$  is a viscosity supersolution of (2.2a)-(2.2b). Let  $\Sigma^-$  be the projection of  $A_{-\lambda_0}^-$  on  $x_N$ -axis. Applying Lemmas on the boundary of  $\Sigma^-$  implies that  $v(x', x_N) = -\lambda_0$  for all  $x_N \in \partial\Sigma^-$ ,  $x' \in \bar{\Omega}'$ . This is a contradiction.  $\square$

We only give the proof of Lemma 3.1. Then we can prove Lemma 3.2. However, we do not give it here.

*Proof of Lemma 3.1.* We may assume that  $K = 0$  since  $w$  plus a constant is still a subsolution when  $w$  is a subsolution. We may also assume that  $M = 0$  by a translation.

We argue by contradiction. Suppose that there would exist  $\zeta' \in D'$  such that  $w(\zeta', 0) < 0 = K$ . The basic strategy for the proof is to find a domain  $E$  in  $D$  and a test function  $\varphi \in C^2(E)$  that satisfies

$$\max_E (w - \varphi) = (w - \varphi)(\hat{x}', \hat{x}_N), \quad (3.1)$$

$$-|\nabla \varphi| \operatorname{div}(\nabla \varphi / |\nabla \varphi|) > 0 \quad \text{at} \quad (\hat{x}', \hat{x}_n) \quad (3.2)$$

for some  $\hat{x} = (\hat{x}', \hat{x}_N) \in E$ . This evidently contradicts the assumption that  $w$  is a subsolution in  $D$ . Our construction of  $\varphi$  and  $E$  reflects the proof of the classical strong maximum principle in [PW], [GT].

*1. Choice of a test function.* Let  $w_0$  be a function on  $D'$  of form

$$w_0(x') = w(x', 0).$$

Since  $w_0$  is upper semicontinuous, there is an open ball  $B_0$  with  $\overline{B_0} \subset D'$  that satisfies

$$\begin{aligned} w_0 &< 0 \quad \text{in} \quad B_0 \quad \text{and} \\ w_0(y') &= 0 \quad \text{for some} \quad y' \in \partial B_0. \end{aligned}$$

This is standard; see e.g. [PW]. (Indeed, we take a curve  $\gamma$  starting from  $\zeta'$  to  $\xi'$  and denote by  $\eta'$  the first point attaining  $w_0 = 0$  on  $\gamma$  starting from  $\zeta'$ . Then there exists a point  $\zeta'_1$  on the arc  $\zeta'\eta'$  such that

$$\zeta'_1 \in B(\eta', d/2) \subset D',$$

where

$$d = \operatorname{dist}(\gamma, \partial D')$$

and  $B(\eta', \sigma)$  denotes the open ball in  $\mathbf{R}^{N-1}$  of radius  $\sigma$  centered at  $\eta'$ . We set

$$r_0 = \sup\{r; w_0(x') < 0 \quad \text{for all} \quad x' \in B(\zeta'_1, r) \subset D'\}$$

so that

$$r_0 < |\zeta'_1 - \eta'| < d/2.$$

If we set  $B_0 = B(\zeta'_1, r_0)$ , then  $B_0$  satisfies all desired properties.)



Let  $B_1$  be a little bit smaller open ball in  $B_0$  such that  $\partial B_0 \cap \partial B_1 = \{y'\}$ . Let  $a$  be the center of  $B_1$  and  $r_1 (< r_0)$  be the radius of  $B_1$ . We take

$$\begin{aligned}\varphi(x', x_N) &= -\varepsilon_1 z(x') - \varepsilon_2 x_N, \\ z(x') &= e^{-\gamma|x'-a|^2} - e^{-\gamma r_1^2}\end{aligned}$$

with positive parameters  $\varepsilon_1, \varepsilon_2$  and  $\gamma$  to be determined later. By definition one observe that

$$\begin{aligned}0 < z(x') < 1 & \quad \text{in } B_1 = B(a, r_1), \\ z(x') &= 0 \quad \text{on } \partial B_1, \\ -1 < z(x') < 0 & \quad \text{outside } \overline{B_1}.\end{aligned}\tag{3.3}$$

2. *Choice of  $\gamma$ .* For each  $\mu = \varepsilon_2/\varepsilon_1$  there is  $\gamma_0 = \gamma_0(\mu)$  such that for  $\gamma \geq \gamma_0$  it holds

$$-|\nabla\varphi| \operatorname{div}(\nabla\varphi/|\nabla\varphi|) > 0 \quad \text{at all } (x', x_N)\tag{3.4}$$

with

$$\frac{r_1}{2} \leq |x' - a| \leq \frac{3r_1}{2}, \quad x_N \in \mathbf{R}.$$

Since

$$-|\nabla\varphi| \operatorname{div}(\nabla\varphi/|\nabla\varphi|) = \varepsilon_1(|\nabla' z(x')|^2 + \mu^2)^{1/2} H(z)$$

with  $H(z) = \operatorname{div}'\{ \nabla' z(x') / (\mu^2 + |\nabla' z(x')|^2)^{1/2} \}$ , it suffices to prove that  $H(z)(x') > 0$  for  $x'$  with  $r_1 \leq 2|x' - a| \leq 3r_1$  when  $\gamma$  is sufficiently large. Here  $\nabla'$  denotes the gradient in  $x'$  and  $\operatorname{div}'$  denotes the divergence in  $x'$ .

Since  $z(x')$  is radial, i.e.,

$$\begin{aligned}z(x') &= g(|x' - a|) \quad \text{with } g(\rho) = e^{-\gamma\rho^2} - e^{-\gamma r_1^2}, \\ H(z) &= \left( \frac{g'}{((g')^2 + \mu^2)^{1/2}} \right)' + \frac{N-2}{\rho} \frac{g'}{((g')^2 + \mu^2)^{1/2}} \Big|_{\rho=|x'-a|}\end{aligned}$$

Since  $g'(\rho) = -2\gamma\rho e^{-\gamma\rho^2}$ ,  $g''(\rho) = -2\gamma e^{-\gamma\rho^2} + 4\gamma^2\rho^2 e^{-\gamma\rho^2}$ , we obtain

$$H(z) = \frac{\{4\mu^2\gamma^2\rho^2 - 2(N-1)\mu^2\gamma - 8(N-2)\gamma^3\rho^2 e^{-2\gamma\rho^2}\} e^{-\gamma\rho^2}}{(4\gamma^2\rho^2 e^{-2\gamma\rho^2} + \mu^2)((g')^2 + \mu^2)^{1/2}}$$

with  $\rho = |x' - a|$ . The quantity in  $\{ \}$  is uniformly positive for  $\rho$ ,  $r_1 \leq 2\rho \leq 3r_1$  provided that  $\gamma$  is sufficiently large say  $\gamma > \gamma_0(\mu)$ .

3. *Choice of the domain  $E$ ,  $\varepsilon_1, \varepsilon_2$ .* Let  $y'$  be the point as in Step 1. By definition

$$w_0 < 0 \quad \text{in} \quad \overline{B_1} \setminus \{y'\} \quad \text{and} \quad w_0(y') = 0.$$

We set  $B_2 = B(y', r_1/2)$ . Since  $r_1 < r_0 < d/2$ ,  $B_2$  is contained in  $D'$ . We take  $\delta > 0$  so small that

$$\partial(B(a, r_1 + \delta)) \cap \partial B_2 \subset B_0.$$

We then divide the boundary of  $B_2$  into two pieces:

$$C'_2 = \partial B_2 \cap \overline{B(a, r_1 + \delta)}, \quad C''_2 = \partial B_2 \setminus \overline{B(a, r_1 + \delta)};$$

clearly  $\partial B_2$  is a disjoint union of  $C'_2$  and  $C''_2$ . Since  $w_0 < 0$  on a compact set  $C'_2$ , there exists a constant  $\ell > 0$  that satisfies  $w_0 \leq -\ell$  on  $C'_2$  by upper semicontinuity of  $w_0$ . Since  $w$  is upper semicontinuous,

$$w \leq -\ell/2 \quad \text{on} \quad C'_2 \times [\alpha', \beta'], \quad [\alpha', \beta'] \subset (\alpha, \beta)$$

for  $\alpha' < 0 < \beta'$  sufficiently close to zero. We first fix  $\alpha' < 0$  since  $|z(x')|$  on  $\overline{B_2}$  is bounded by 1 by (3.3), we take  $\mu > (-\alpha')^{-1}$  so that

$$\sup\{z(x'); x' \in B_2\}(-\alpha')^{-1} < \mu \tag{3.5}$$

for all  $\gamma > 0$ . We fix  $\gamma$  with  $\gamma > \gamma_0(\mu)$  so that (3.4) holds. We then take  $\beta'$  smaller so that

$$-\sup\{z(x'); x' \in C''_2\}/\beta' > \mu. \tag{3.6}$$

We set

$$\begin{aligned} \sigma_1 &= \sup\{w(x', x_N); x' \in C'_2, \alpha' < x_N < \beta'\}, \\ \sigma_2 &= \sup\{w(x', \beta'); x' \in \overline{B_2}\}. \end{aligned}$$

By definition of  $C'_2$  and  $M = 0$  we see that  $\sigma_1 \leq -\ell/2$ ,  $\sigma_2 < 0$ . Choose  $\varepsilon_1, \varepsilon_2$  sufficiently small so that

$$\max\{\sigma_1, \sigma_2\} + \varepsilon_1 + \varepsilon_2 \beta' < 0 \tag{3.7}$$

keeping  $\mu = \varepsilon_2/\varepsilon_1$ . We take  $E = B_2 \times (\alpha', \beta')$  and fix  $\alpha', \mu, \gamma, \beta', \varepsilon_1, \varepsilon_2$  satisfying (3.5)-(3.7) with  $\gamma > \gamma_0(\mu)$ .

4. *Completion of the proof.* To show (3.1) it suffices to prove

$$\max_{\partial E} (w - \varphi) < 0 \quad (3.8)$$

since  $(w - \varphi)(y', 0) = 0$  and  $(y', 0) \in E$ . We divide  $\partial E$  into four pieces

- (a)  $x' \in C'_2$  and  $\alpha' < x_N < \beta'$ ,
- (b)  $x' \in C''_2$  and  $\alpha' < x_N < \beta'$ ,
- (c)  $x' \in \overline{B}_2$  and  $x_N = \alpha'$ ,
- (d)  $x' \in \overline{B}_2$  and  $x_N = \beta'$ .

On the part (a) because of a bound  $w \leq -\ell/2$  we conclude  $w - \varphi$  is negative if  $\varepsilon_1, \varepsilon_2$  is taken by (3.7); note that  $|z|$  is bounded independent of  $\gamma$  by (3.3). On the part (b) by (3.3)

$$\sup\{z(x'); x' \in C''_2\} < 0.$$

The negativity of  $w - \varphi$  follows from (3.6). On the part (c) the negativity of  $w - \varphi$  follows from (3.5). On the part (d) since  $\sigma_2 < 0$ , (3.7) implies the negativity of  $w - \varphi$ . Thus we have proved (3.8). Since (3.4) holds on  $B_2 \times \mathbf{R}$ , we get desired  $\varphi$  and  $E$  satisfying (3.1) and (3.2).  $\square$

*Remark 3.3.* Our Theorem 2.5 as well as Lemmas 3.1 and 3.2 applies more general equation than (2.2a). We may replace (2.2a) by

$$F(\nabla u, \nabla^2 u) = 0 \quad (3.9)$$

with  $F$  satisfying

- (i)  $F : (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \rightarrow \mathbf{R}$  is continuous and geometric in the sense of [CGG].
- (ii)  $F(p, O) = 0$  for all  $p \in \mathbf{R}^N \setminus \{0\}$ .
- (iii) For each  $\lambda_0 > 0$  there exists  $N_0 > 0$  such that if  $\lambda_{\max}(Q_{\bar{p}}(X)) \leq \lambda_0$  and  $\lambda_{\min}(Q_{\bar{p}}(X)) \leq -N_0$  (resp.  $\lambda_{\min} \geq -\lambda_0$ ,  $\lambda_{\max} \geq N_0$ ) then  $F(p, Q_{\bar{p}}(X)) > 0$  (resp.  $< 0$ ) for all  $X \in \mathbf{S}^N$  and  $p \in \mathbf{R}^N \setminus \{0\}$ , where  $Q_{\bar{p}}(X) = (I - \bar{p} \otimes \bar{p})X(I - \bar{p} \otimes \bar{p})$  with  $\bar{p} = p/|p|$ .

Here  $\mathbf{S}^N$  denotes the space of all real symmetric matrices and  $\lambda_{\min}(Y)$  and  $\lambda_{\max}(Y)$  are the smallest and the largest eigenvalues of  $Y \in \mathbf{S}^N$ , respectively. Even if (2.2a) is replaced by (3.9) the proof of Lemma 3.1 is the same except step 2 where we have to replace (3.4) by

$$F(\nabla \varphi, \nabla^2 \varphi) > 0 \quad \text{at all } (x', x_N) \quad (3.4')$$

satisfying  $r_1 \leq 2|x' - a| \leq 3r_1$ ,  $x_N \in \mathbf{R}$ . To prove (3.4') for large  $\gamma \geq \gamma_0(\mu)$  the property (iii) is invoked. For example,

$$F(p, X) = -\text{trace}\{A(-\bar{p})Q_{\bar{p}}(X)\}$$

satisfies the above conditions (i)–(iii), where  $A(\bar{p})$  is a given matrix in  $S^N$  and positive definite for  $p \neq 0$ . This  $F$  appear when we study a level set equation of the anisotropic mean curvature flow equation (for the anisotropic mean curvature equation see e.g. [Gur] and for its level set equation see e.g. [CGG].) Here we shall check the conditions (i)–(iii). For (i) and (ii) we can check easily. It remains to show (iii). We may assume that  $Q_{\bar{p}}(X)$  is a diagonal matrix. Let  $A(-\bar{p}) = (a_{ij})$  and let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be eigenvalues of  $Q_{\bar{p}}(X)$  with  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ . Then we see

$$-\text{trace}\{A(-\bar{p})Q_{\bar{p}}(X)\} = -\sum_{i=1}^N \lambda_i a_{ii}.$$

From the assumption  $\lambda_N = \lambda_{\max}(Q_{\bar{p}}(X)) \leq \lambda_0$  and  $\lambda_1 = \lambda_{\min}(Q_{\bar{p}}(X)) \leq -N_0$  we observe that

$$-\sum_{i=1}^N \lambda_i a_{ii} \geq -\lambda_1 a_{11} - \sum_{i=2}^N \lambda_0 a_{ii}.$$

If  $|\lambda_1|$  is sufficiently large then the condition (iii) holds. A similar remark applies Lemma 3.2. (Geometricity is not invoked for Lemmas 3.1 and 3.2.) To extend Theorem 2.5 for (3.9) we notice that properties (i)–(iii) are invariant under translation in space independent variables and order-preserving change of the dependent variable of (3.9); (i)–(iii) are invariant under multiplication with  $-1$  to the dependent variable by taking  $\tilde{F}(p, X) = F(-p, -X)$ .

This extended theory applies level set equations of anisotropic mean curvature flow equations (see e.g. [Gur]) provided that the Frank diagram of interfacial energy is strictly convex in the sense that its all (inward) principal curvatures are positive.

*Remark 3.4.* Recently, a strong maximum principle for degenerate elliptic equations in viscosity sense was established by Bardi and Da Lio. Although they study fully nonlinear partial differential equation of the form

$$F(x, u, \nabla u, \nabla^2 u) = 0,$$

here we only explain thier results on the strong maximum principle for the equation (3.9). Let  $F : (\mathbf{R}^N \setminus \{0\}) \times \mathbf{S}^N \rightarrow \mathbf{R}$  be continuous and be lower semicontinuous on

$\mathbf{R}^N \times \mathbf{S}^N$ . Assume that  $F$  is degenerate elliptic, i.e.,

$$F(p, X) \leq F(p, Y) \quad \text{if } X \geq Y \quad \text{and for all } p \neq 0.$$

Moreover, they assume two properties on  $F$ . One is *the nondegeneracy property*, that is, there exist  $\gamma_0 > 0$  such that

$$F(\nu, I - \gamma\nu \otimes \nu) > 0 \quad \text{for all } \gamma > \gamma_0, \nu \neq 0. \quad (3.10)$$

The other is *the scaling property*, that is, there exist a function  $\varphi > 0$  such that

$$F(\xi s, \xi X) \geq \varphi(\xi)F(s, X) \quad \text{for all } \xi > 0, s \in [-1, 0]. \quad (3.11)$$

There are many equations satisfying the above conditions. For example, the minus  $p$ -Laplacian, the minus  $\infty$ -Laplacian and the graph minimal surface equation. However, the level set minimal surface equation does not satisfy the condition (3.10). Generally, geometric equations do not fulfill it. Their proof of the strong maximum principle reflects the proof of the classical strong maximum principle in [PW], [GT] as same as our Lemma 3.1.

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